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Gauged Q ball in a piecewise parabolic potential

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Abstract

Q -ball solutions are considered within the theory of a complex scalar field with a gauged $U(1)$ symmetry and a parabolic-type potential. In the thin-walled limit, we show explicitly that there is a maximum size for these objects because of the repulsive Coulomb force. The size of the Q ball will increase with decreasing local minimum of the potential. And when the two minima degenerate, the energy stored within the surface of the Q ball becomes significant. Furthermore, we find an analytic expression for a gauged Q ball, which is beyond the conventional thin-walled limit.

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1. Introduction

Recently, Theodorakis introduced a piecewise parabolic potential for a complex scalar field ϕ and he showed that it admits stable Q -ball solutions [1]. These solutions have been found analytically, unlike the case of polynomial potentials. The Q ball is a non-topological soliton, and its stability depends on whether charge can be lost through emission of charged particles [2, 3]. The generation of the Q ball [4] and the possibility of phase transition precipitated by soliton synthesis [5] have been actively studied. Later work also examined Q -stars [6] and D-stars [7, 8] with interesting astrophysical implications. The non-topological soliton can appear in a local $U(1)$ invariant theory [9–11].

Following Coleman [3], at large distances from the Q ball the field should approach the vacuum solution $\phi = 0$, if the charge is to be finite. It turns out that Q balls exist in a potential $V(\phi)$, for large enough values of the charge Q , if the function $2V/|\phi|^2$ has a minimum of ϕ . The simplest form of the potential is a polynomial one where a negative $|\phi|^4$ term is needed to make the potential dip below $|\phi|^2/2$, and a positive $|\phi|^6$ term to show that the potential is bounded from below. However, these potentials give a difficult task for solving equations of motion. It would be interesting to obtain analytic solutions of the Q ball. Fortunately, there is

no reason, at least at the classical level, to restrict ourselves to polynomial potentials. In fact, Theodorakis [1] has examined a piecewise parabolic potential as follows:

$$U(|\phi|) = \frac{1}{2} [|\phi|^2 + \varepsilon(1 - |\phi|) - \varepsilon|1 - |\phi| |]. \quad (1)$$

In the $1 < \varepsilon < 2$ case, the global minimum is at $|\phi| = 0$, but there is a local minimum at $|\phi| = \varepsilon$, the potential being positive everywhere. In this paper, we consider the gauged Q balls where the potential is parabolic-type one. Hitherto, these were restricted to numerical methods or the thin-walled approximation in the study of the gauged Q ball. In the thin-walled approximation, we find that the radius of the Q ball is increased with the decline of the local minimum of the potential. And when the two minima are degenerated, the surface term will play an important role in the energy functional. Using an iteration method, we have shown an explicit solution of the Q ball with gauge interaction beyond the thin-walled limit in this paper. Furthermore, we can give the asymptotic solutions order by order in principle.

2. Qualitative properties of the solution

We consider the theory with a complex scalar field ϕ coupled to a $U(1)$ gauge field A_μ . The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} |(\partial_\mu - ieA_\mu)\phi|^2 - U(|\phi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (2)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $U(|\phi|)$ is the parabolic-type potential (1). The conserved current and the energy-momentum tensor which induce the conserved quantities are

$$j_\mu = \frac{1}{2} i [\phi(\partial_\mu + ieA_\mu)\bar{\phi} - \bar{\phi}(\partial_\mu - ieA_\mu)\phi] \quad (3)$$

and

$$T_{\mu\nu} = \frac{1}{2} [(\partial_\mu - ieA_\mu)\phi(\partial_\nu + ieA_\nu)\bar{\phi} + (\partial_\nu - ieA_\nu)\phi(\partial_\mu + ieA_\mu)\bar{\phi}] - F_{\mu\sigma} F_\nu^\sigma - \mathcal{L}\eta_{\mu\nu} \quad (4)$$

respectively. To find the solutions of the Q ball, we use the following ansatz:

$$\phi = f(r) e^{i\omega t} \quad (5)$$

and

$$A_\mu = A_0(r)\delta_{\mu 0} \quad (6)$$

where we assume $\omega > 0$ for definiteness. The lowest energy state will have no electric currents and therefore no magnetic fields. The spatial components of the gauge potential are zero as there is no magnetic field. We choose a boundary condition $A_0 \rightarrow 0$ as $r \rightarrow \infty$.

Using the above configurations, the Lagrangian becomes

$$L = 4\pi \int dr r^2 \left[-\frac{1}{2} f'^2 + \frac{1}{2e^2} g^2 + \frac{1}{2} g^2 f^2 - U(f) \right] \quad (7)$$

where $g = \omega - eA_0(r)$, and a prime denotes d/dr . The Noether charge associated with the $U(1)$ symmetry becomes

$$Q = \int d^3x g f^2. \quad (8)$$

When we insert equations (5) and (6) in the general field equation deduced from equation (2) we find the following equations of motion:

$$f'' + \frac{2}{r}f' + fg^2 = f - \varepsilon \quad r \leq R \quad f \geq 1 \quad (9)$$

$$f'' + \frac{2}{r}f' + fg^2 = f \quad r > R \quad f < 1 \quad (10)$$

$$g'' + \frac{2}{r}g' = e^2 f^2 g. \quad (11)$$

The fact that the field equations (9)–(11) are nonlinear implies that the analytic solutions that will be found will be approximate. To gain insight into the Q ball for $e \neq 0$, we show some qualitative properties of the solution. From (8) and (11) we have

$$g(r) \rightarrow \omega - \frac{e^2 Q}{4\pi r} \quad (12)$$

as $r \rightarrow \infty$. Then equation (10) can be reduced to the asymptotic form

$$\frac{1}{r}(rf)'' + (\omega^2 - 1)f = 0. \quad (13)$$

For the approximate equation (13), we have a solution as follows:

$$f(r) = f_0 \exp(-\sqrt{1 - \omega^2}r)/r. \quad (14)$$

Clearly, a necessary condition for the existence of a solution is $\omega < 1$. Additionally, for the solution to be well behaved at the origin, f' and g' should approach zero at least faster than r for $r \rightarrow 0$. By using the boundary conditions of f and g the energy functional can be written as

$$E = \frac{1}{2}\omega Q + 4\pi \int r^2 dr \left[\frac{1}{2}f'^2 + V(f) \right]. \quad (15)$$

For the $e \neq 0$ case, we expect that the energy will be increased over the $e = 0$ case due to Coulomb repulsion, with the Coulomb energy becoming more important as Q becomes large. Therefore, if $\partial E/\partial Q > 1$, we must consider that some charge can be put into the interior region of the Q ball and some can be put in free particles. As discussed above, there exists a Q_{\min} and a Q_{\max} , so that when $Q > Q_{\min}$ the soliton is quantum mechanically stable and when $Q > Q_{\max}$ the lowest energy state of the system is composed of a soliton with charge Q_{\max} together with free particles carrying charge $Q - Q_{\max}$. In the $e = 0$ theory, since $Q_{\max} \rightarrow \infty$, the condition $Q_{\min} < Q_{\max}$ is always satisfied.

3. Thin-walled approximation

If we choose a continuous ansatz for the scalar field, the most convenient one might be the piecewise ansatz

$$f(r) = \begin{cases} \tilde{f} & r < R \\ \tilde{f} - \frac{\tilde{f}}{\delta}(r - R) & R \leq r \leq R + \delta \\ 0 & r > R + \delta \end{cases} \quad (16)$$

where δ is the width of the wall and \tilde{f} is a constant. But unfortunately, this ansatz, unlike the $e = 0$ case, does not satisfy the equations of motion of the scalar field. Therefore, one must

choose the thin-walled approximation in which the spatial configuration of the scalar field $f(r)$ can be considered as a step function.

In the thin-walled approximation, the spatial configuration of the scalar field $f(r)$ can be considered as a step function,

$$f(r) = \tilde{f}\theta(R - r). \quad (17)$$

Although the gradient of f in the thin surface is a delta function, and hence the associated density of kinetic energy is infinite, the surface term in the integral of energy is finite. In particular, in the case of $\varepsilon < 2$, the contribution of the kinetic energy term to the integral of energy is negligible. Therefore, the constant \tilde{f} in equation (17) is determined by minimizing the energy functional. In this case, we can solve equation (11)

$$g(r) = \frac{(\omega - e^2 Q/4\pi R)R \sinh(e\tilde{f}r)}{\sinh(e\tilde{f}R) r} \quad \text{for } r \leq R \quad (18)$$

$$g(r) = \omega - \frac{e^2 Q}{4\pi r} \quad \text{for } r > R \quad (19)$$

where the gauge is chosen so that $A_0 \rightarrow 0$ for $r \rightarrow \infty$. Inserting the above solution into equation (8), we have

$$\omega = \frac{e^2 Q}{4\pi R} \left[1 - \frac{\tanh(e\tilde{f}R)}{e\tilde{f}R} \right]^{-1}. \quad (20)$$

The total energy of the ball is reduced to

$$E = \frac{e^2 Q^2}{8\pi R} \left[1 - \frac{\tanh(e\tilde{f}R)}{e\tilde{f}R} \right]^{-1} + \frac{4\pi}{3} U(\tilde{f}) R^3. \quad (21)$$

From $\partial E/\partial R|_{R=R_*} = 0$, the radius R_* of the ball is determined by

$$4\pi \sqrt{2U(\tilde{f})/\tilde{f}^2} R \left[\frac{e\tilde{f}R}{\tanh(e\tilde{f}R)} - 1 \right] = e^2 Q. \quad (22)$$

It is a formidable task to find exact solutions of the transcendental equation (22). We have to use $e\tilde{f}R \ll 1$ approximation [9]. Equation (21) can be reduced to the following form:

$$E = Q \left[\frac{2U(\tilde{f})}{\tilde{f}^2} \right]^{1/2} \left[1 + \frac{(\frac{3}{4\pi}\tilde{f}Q)^{2/3} (\tilde{f}^2/2U(\tilde{f}))^{1/3}}{5} \right]. \quad (23)$$

Minimizing the energy with respect to \tilde{f} and $Q(R)$ we find that

$$\tilde{f}_* = 2 - \frac{8e^2}{5\varepsilon} \left[\frac{Q^2(1 - \frac{1}{2}\varepsilon)^2}{3\pi} \right]^{1/3} \quad (24)$$

$$R_* = \left[\frac{3Q}{8\pi\sqrt{2(2-\varepsilon)}} \right]^{1/3} \left[1 + \frac{1}{45} \left(\frac{3e^3 Q}{\pi\sqrt{2(2-\varepsilon)}} \right)^{2/3} \right] \quad (25)$$

and

$$E_* = Q \sqrt{1 - \frac{\varepsilon}{2}} \left\{ 1 + \frac{1}{5} \left(\frac{3e^3 Q}{\pi\sqrt{2(2-\varepsilon)}} \right)^{2/3} \right\}. \quad (26)$$

For a fixed charge Q , we see that both the radius and the energy are larger than the ungauged one. Furthermore, from the condition $\partial E/\partial Q|_{Q=Q_{\max}} = 1$, one can obtain the upper bound on the total charge Q_{\max} as

$$Q_{\max} = \frac{2\pi}{3e^2} \left[5 \left(\sqrt{\frac{2}{2-\varepsilon}} - 1 \right) \right]^{3/2} \sqrt{\frac{2-\varepsilon}{2}}. \quad (27)$$

When $Q > Q_{\max}$ the lowest energy state of the system is composed of a Q ball with charge Q_{\max} together with free particles carrying charge $Q - Q_{\max}$. Here note that the inconsistency of the approximation $efR \ll 1$ when ε approaches 2. In the $\varepsilon = 2$ case, we must consider the surface term with a finite width. If $efR \gg 1$, as will be discussed later, the energy can be written approximately as

$$E = \frac{e^2 Q^2}{8\pi R} + 4\pi \varepsilon R^2. \quad (28)$$

From $\partial E/\partial R|_{R=R_*} = 0$, the radius of the Q ball is given by

$$R_* = \left(\frac{e^2 Q^2}{128\pi^2} \right)^{1/3} \quad (29)$$

and

$$E_* = E(R_*) = \frac{3(eQ)^{4/3}}{2^{2/3}\pi^{1/3}}. \quad (30)$$

The upper bound on charge Q_{\max} is also given by

$$Q_{\max} = \frac{\pi}{2e^4}. \quad (31)$$

For simplicity, we write f_* , R_* and E_* as f , R and E hereafter.

4. Beyond the thin-walled limit

We may use the successive approximation method of differential equations for a higher-order approximation of the Q -ball solution. Using $rg(r) = e^{u(r)}$ and $v(r) = u'(r)$, equation (11) can be reduced to

$$v' = e^2 f^2 - v^2 \quad (32)$$

with the initial condition $v = v_0$ for $r = r_0$. It can be written in the form

$$v = v_0 + \int_{r_0}^r (e^2 f^2 - v^2) dr. \quad (33)$$

Substituting the thin-walled solution $v_1(r)$ instead of v into the right-hand term, we obtain a new function v_2 different from v_1 , unless v_1 is an exact solution of equation (32). Substituting v_2 instead of v into the right-hand side of equation (32), we obtain a function v_3 and so on. The sequence $v_1, v_2, \dots, v_n, \dots$ obtained in this way is convergent in a certain interval containing r_0 to the desired solution of the given equation, provided that the assumptions of Cauchy's existence theorem hold. In principle, we can find the exact solution $g(r)$ if $f(r)$ is known. This method is also called the iteration method (see figure 1).

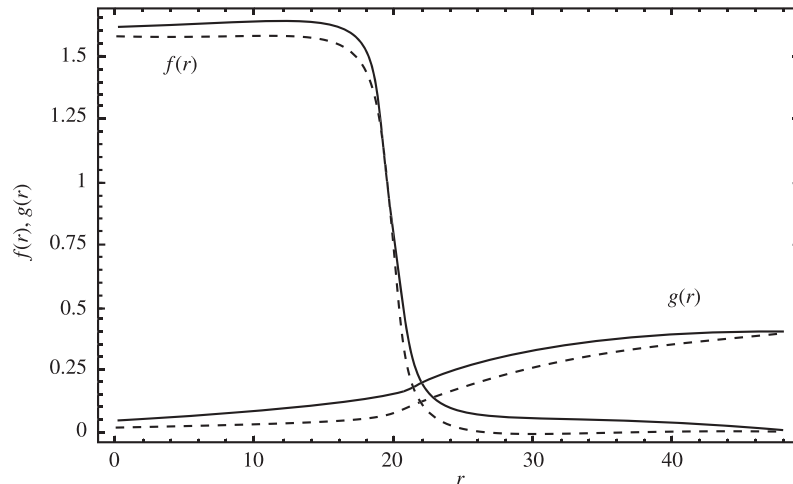


Figure 1. Plot of the scalar field $f(r)$ and the gauged field $g(r)$ as a function of r . The full curves are numerical results and the broken curves are our iteration solution. The plots are drawn at $e = 0.1$, $Q = 10\,000$ and $\varepsilon = 1.8$.

In the weak-coupling situation $e \ll 1$ and $g(r) = \omega$, we have solutions of equation (9) and (10) as follows:

$$f = a + b \frac{\sinh(\nu r)}{r} \quad r \leq R \tag{34}$$

$$f = \frac{R}{r} e^{\nu(R-r)} \quad r > R \tag{35}$$

where $\nu = \sqrt{1 - \omega^2}$, $a = \varepsilon/\nu^2$ and $b = (1 - \varepsilon/\nu^2)R/\sinh \nu R$. For thin-walled limit, we have $\tilde{f} = a + b\nu$ in equation (19). Equivalently, we have

$$v_1(r) = e(a + b\nu) \coth[e(a + b\nu)r] \quad r \leq R \tag{36}$$

$$v_1(r) = \left(r - \frac{e^2 Q}{4\pi\omega}\right)^{-1} \quad r > R. \tag{37}$$

Substituting $v_1(r)$ instead of v to equation (33), we obtain high-order approximation

$$v_2 = e^2 a^2 r + 2e^2 ab \operatorname{Shi}(\nu r) + e^2 b^2 \nu \operatorname{Shi}(2\nu r) - \frac{e^2 b^2 \sinh^2(\nu r)}{2r} + e(a + b\nu) \{ \coth[e(a + b\nu)r] - e(a + b\nu)r \} + C \quad r \leq R$$

$$v_2 = \left(r - \frac{e^2 Q}{4\pi\omega}\right)^{-1} - \frac{e^2 c^2 e^{-2\nu r}}{r} - 2e^2 R^2 \nu \exp(2\nu R) \operatorname{Ei}(-2\nu r) \quad r > R \tag{38}$$

where $\operatorname{Shi}(x)$ is the hyperbolic-sine-integral function and $\operatorname{Ei}(x)$ is the exponential-integral function. Using the relation

$$\operatorname{Ei}(-2\nu r) = -e^{2\nu r} \int_1^\infty \frac{1}{2\nu r + \ln t} \frac{dt}{t^2} \tag{39}$$

one can easily find that $v(r)$ satisfies the boundary condition at spatial infinity $v(\infty) = 0$. The integral constant C can be fixed by continuity at $r = R$. Therefore, we have

$$g_2 = \left(\omega - \frac{e^2 Q}{4\pi R}\right) \frac{\sinh[e(a + b\nu)r]}{\sinh[e(a + b\nu)R]} \left(\frac{r}{R}\right)^{\frac{1}{2}e^2 b^2 - 1} \exp\left\{\frac{e^2}{2}(2ab\nu + b^2\nu^2)(R^2 - r^2)\right\}$$

$$\begin{aligned}
& +2e^2 ab[r \operatorname{Shi}(\nu r) - R \operatorname{Shi}(\nu R)] + \frac{2}{\nu} e^2 ab[\cosh(\nu R) - \cosh(\nu r)] \\
& +e^2 b^2 \nu[r \operatorname{Shi}(2\nu r) - R \operatorname{Shi}(2\nu R)] + e^2 b^2[\cosh(2\nu R) - \cosh(2\nu r)] \\
& +e^2 R^2 \nu + 2e^2 R^2 \nu \exp(2\nu R) \int_R^\infty \operatorname{Ei}(-2\nu r) dr \Big\} \quad r \leq R \quad (40)
\end{aligned}$$

$$g_2 = \left(\omega - \frac{e^2 Q}{4\pi R} \right) \exp \left\{ e^2 R^2 \nu \exp(2\nu R) \left[\operatorname{Ei}(-2\nu r) + \int_r^\infty \operatorname{Ei}(-2\nu r) dr \right] \right\} \quad r > R. \quad (41)$$

Equations (34), (35), (40) and (41) are the solution of a high-order approximation beyond the thin-walled limit. Our approximate analytic solution is in good agreement with the numerical results [9], which have been obtained by many authors. The study on the Q ball in a parabolic potential, as a toy model, can enable us to understand the Q ball analytically, which provides more information than a numerical study. In principle, by continuing the procedures mentioned above, we can obtain the approximate solution with any high-order accuracy. It is obvious that this method can also be employed in investigating other spherically symmetric solitons, and the corresponding results will appear elsewhere.

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